Journal of Statistical Physics, Vol. 58, Nos. 3/4, 1990

Branched Polymers in a Wedge Geometry in Three Dimensions

D. S. Gaunt[1] **and S. A. Colby**[1,2]

Received July 14, 1989

We investigate the statistical and dimensional properties of uniform star polymers attached by the branching vertex of degree f in a wedge geometry in three dimensions and described by the wedge angles θ and ϕ. We show that the growth constant is equal to μ^f, where μ is the self-avoiding walk limit. The f and (θ, ϕ) dependences of the corresponding critical exponent $\gamma_f(\theta, \phi)$ are studied using Monte Carlo techniques. In the case $f = 1$, our results are compared with existing predictions obtained from series expansion and renormalization group methods. We have also estimated the amplitudes for the mean square radius of gyration and the mean square end-to-end branch length. Our results for the ratio of the mean square radius of gyration of an f-star to that of a linear polymer of the same degree of polymerization attached in a similar wedge, and the analogous ratio for the mean square end-to-end branch length, are consistent with these ratios being lattice-independent quantities.

KEY WORDS: Uniform star polymers; three-dimensional wedge geometry; growth constant; Monte Carlo; critical exponents.

1. INTRODUCTION

Over the last few years, there has been considerable interest in the excluded-volume effect in uniform star-branched polymers. These structures have f branches meeting at a common branching point and have the same number of monomers in each of their branches. Theoretical work on the statistics and dimensions of uniform stars includes a scaling theory,[1,2] a renormalization group (RG) treatment,[3-5] and the use of conformal invariance arguments.[6] Numerical results for a lattice version of this model have

[1] Department of Physics, King's College, London WC2R 2LS, United Kingdom.
[2] Present address: Easams Ltd, Camberley, Surrey GU16 5EX, United Kingdom.

0022-4715/90/0200-0539$06.00/0

been obtained by exact enumeration and Monte Carlo methods[7-10] and are in quite good agreement with the theoretical predictions.

Suppose that the number of uniform stars with n monomers in each of the f branches is $s_n(f)$. One expects that asymptotically $(n \to \infty)$

$$s_n(f) \sim n^{\gamma_f - 1} \lambda(f)^n \tag{1}$$

where $\lambda(f)$ is the growth constant for f-stars and γ_f is the corresponding critical exponent. It can be shown[8] that the growth constant, whose existence can be proved rigorously, is given by

$$\lambda(f) = \mu^f \tag{2}$$

where μ is the growth constant for self-avoiding walks (SAWs). In $d=2$ dimensions, the exponent γ_f has been calculated exactly using conformal invariance arguments.[6] For $d=3$, γ_f is known by RG methods[3-5] through $O(\varepsilon)$, and has been estimated numerically for small f using exact enumeration and Monte Carlo data.[8]

Recently, attention has turned to uniform stars in confined geometries. For example, the growth constants $\lambda(f, L) = [\mu(f, L)]^f$ have been studied in parallel-sided slabs and slits of width L. For $d \geqslant 3$, $\mu(f, L)$ is independent of f and equal to $\mu(L)$, the growth constant of a SAW with the same geometrical constraint.[11] When $d=2$, however, the situation is quite different. It appears then that $\mu(f, L)$ depends on f and is strictly less than $\mu(L)$.[11,12]

In earlier work,[13] we have examined the behavior of f-stars when confined in a wedge geometry in two dimensions. Let the number of uniform f-stars having n bonds in each of the f branches and attached by a vertex of degree $a=1$ or f at the apex of a wedge of angle α be $s_n(\alpha; f, a)$. One expects in analogy with (1) that

$$s_n(\alpha; f, a) \sim n^{\gamma_{f,a}(\alpha) - 1} [\lambda(\alpha; f, a)]^n \tag{3}$$

It is easy to prove, using the methods of Hammersley and Whittington[14] and Chee and Whittington,[11] that

$$\lambda(\alpha; f, a) = \mu^f \tag{4}$$

where μ is the growth constant for SAWs on the parent lattice. Duplantier and Saleur[15] have made use of conformal invariance arguments to calculate values of $\gamma_{f,a}(\alpha)$ in two dimensions and our numerical results[13] are in good agreement with these predictions. This provides some support for the conformal invariance assumptions for these systems.

We have also investigated[13] the mean square end-to-end lengths of the branches and the mean square radius of gyration of stars in a two-dimensional wedge geometry and discussed their sensitivity to the geometry and mode of attachment. In all cases, we found that the corresponding critical exponent ν is equal to the bulk SAW value.

In the present paper, we extend the work of Colby *et al.*[13] to uniform f-stars in three-dimensional wedges. These are defined by three intersecting planes all passing through the origin. Plane I is the (x, y) plane. Plane II is a plane containing the x axis with an angle θ between normals to planes I and II. Plane III is a plane containing the y axis with an angle ϕ between normals to planes I and III. A wedge defined in this way will be referred to as (θ, ϕ)-wedge. For example, a $(\pi/2, \pi/2)$-wedge corresponds to an octant (or cube corner), while an alternative realization of a 1/8-space is a $(\pi/4, \pi)$-wedge. More generally, a (θ, π)-wedge generates a $\theta/2\pi$-space which is described most simply in cylindrical polar coordinates. Clearly, then, a 1/2-space corresponds to a (π, π)-wedge.

We have used an inversely restricted Monte Carlo approach[16] to generate data on the simple cubic (SC) and diamond (D) lattices. The success rate for Monte Carlo growth in three dimensions is much higher than in two dimentions, e.g., about 95% for an $n = 50$ SAW in a half-space on the simple cubic lattice, as compared with 50% on the square lattice. Consequently, we have been able to use a much smaller number of trials than was possible in two dimensions to generate a sample of similar size. Thus, for each kind of wedge, we have made 250,000 trials for SAWs and betweèn 750,000 and 3.15×10^6 trials for various types of f-star ($f > 1$).

For all f, the number of stars attached at the apex of the wedge (i.e., the origin) is expected to have the asymptotic behavior

$$s_n(\theta, \phi; f) \sim n^{\gamma_f(\theta, \phi) - 1}[\lambda(\theta, \phi; f)]^n \tag{5}$$

In this equation, which is analogous to (3), we have suppressed the parameter a, since, in all cases, we have only considered stars attached by the vertex of degree f. This allows us to use a more time efficient method of collecting data, namely grow a set of stars, calculate the required quantities, and then grow a set of larger stars by simply adding bonds to the ends of the stars we already had. We first sketch a proof that the growth constant $\lambda(\theta, \phi; f)$ is given by the analogue of (4), namely

$$\lambda(\theta, \phi; f) = \mu^f \tag{6}$$

where again μ is the growth constant for SAWs on the parent lattice.

2. GROWTH CONSTANT OF UNIFORM STARS IN A (θ, φ) WEDGE

Let the number of f-stars in a diverging wedge Ω be denoted by $s_n(\Omega; f)$. Such a star can be constructed by attaching k stars k stars of degrees $f_1, f_2, ..., f_k$ in disjoint diverging wedges Ω_1, $\Omega_2, ..., \Omega_k$, subject to the constraints

$$\sum_{i=1}^{k} f_i = f, \qquad \bigcup_{i=1}^{k} \Omega_i = \Omega \tag{7}$$

This construction will generate all of the f-stars with sets of branches confined to the disjoint wedges. Hence, we obtain the lower bound

$$s_n(\Omega; f) \geqslant \prod_{i=1}^{k} s_n(\Omega_i; f_i) \tag{8}$$

To derive an upper bound, we concatenate k' stars of degrees $f'_1, f'_2, ..., f'_k$ in the wedge Ω. This construction generates all f-stars with mutually-avoiding branches, as well as some others with intersecting branches. Hence,

$$s_n(\Omega; f) \leqslant \prod_{i=1}^{k'} s_n(\Omega; f'_i), \qquad \sum_{i=1}^{k'} f'_i = f \tag{9}$$

If we choose $k = k' = f$ and $f_i = f'_i = 1$ for all i, then we may rewrite the bounds in terms of the number c_n of SAWs in various wedges, i.e.,

$$\prod_{i=1}^{f} c_n(\Omega_i) \leqslant s_n(\Omega; f) \leqslant \prod_{i=1}^{f} c_n(\Omega) \tag{10}$$

However, we know from the work of Hammersley and Whittington[14] that the growth constant of a SAW in any wedge of the type used here is the same as that of a SAW on the parent lattice. This observation, together with (10), establishes that

$$\lim_{n \to \infty} n^{-1} \log s_n(\Omega; f) = f \log \mu \tag{11}$$

Hence, the growth constant is given by (6). Our Monte Carlo data are in complete agreement with this result.

We note that the rigorous bounds (8) and (9) imply the following inequalities between the corresponding critical exponents (assuming that they exist):

$$\sum_{i=1}^{k'} \gamma_{f_i'}(\Omega) + k - k' \geqslant \gamma_f(\Omega) + k - 1 \geqslant \sum_{i=1}^{k} \gamma_{f_i}(\Omega_i) \tag{12}$$

subject to the constraints in (7) and (9). Discussion of these inequalities is deferred until Section 4.

3. MONTE CARLO CALCULATIONS FOR SAWs

We begin by considering the particular case $f=1$, corresponding to SAWs, since this is the only case where we can make contact with earlier work. Data have been obtained for six different types of wedge, including (θ, π)-wedges with $\theta = \pi/4$, $\pi/2$, π (i.e., the half-plane) and $3\pi/2$, as well as $(\pi/2, \pi/2)$- and $(3\pi/2, 3\pi/2)$-wedges, i.e., "internal" and "external" cube corners, respectively. If $s_n(\theta, \phi; 1)$ behaves as in (5) and (6), we note that a plot of $\ln[s_n(\theta, \phi; 1)/\mu^n]/\ln n$ against $1/\ln n$ will approach $[\gamma_1(\theta, \phi) - 1]$ linearly as $n \to \infty$. Typical plots are shown in Fig. 1 and our best estimates are summarized in Table I.

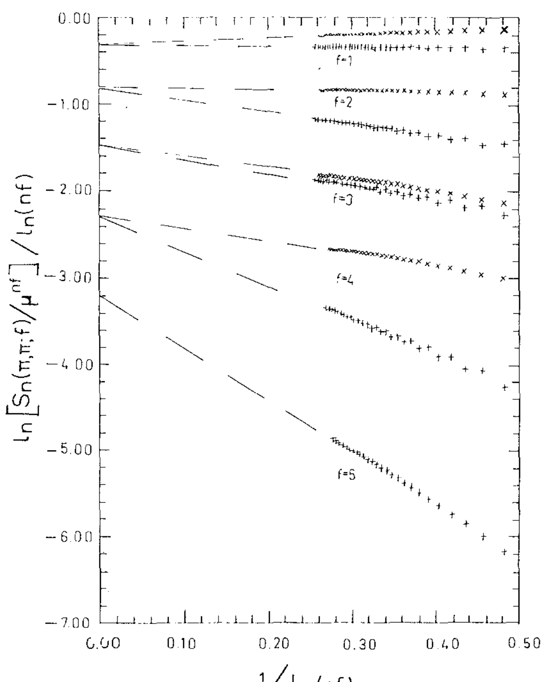

Fig. 1. Monte Carlo estimates of $\gamma_f(\pi, \pi) - 1$ for the diamond ($\times$) and simple cubic ($+$) lattices. The largest error bars are about equal to the size of the symbols.

Many workers have estimated the exponent $\gamma_1(\pi, \pi)$ for a half-space. Series estimates include 0.708 ± 0.008,[17] 0.700 ± 0.005,[18] 0.70 ± 0.02,[19] 0.71 ± 0.02,[20] and 0.676 ± 0.009.[21] In addition, there is a Monte Carlo estimate of 0.69 ± 0.01[22] and an $O(\varepsilon^2)$-expansion estimate of 0.695.[23] Our Monte Carlo result (0.703 ± 0.005) is consistent with all previous estimates except the series estimate of Guttmann and Torrie,[21] where the uncertainties just fail to overlap. We note that all the estimates of $\gamma_1(\pi, \pi)$ satisfy the exact inequality[17]

$$\gamma_1(\pi, \pi) \leqslant (\gamma + 1)/2$$

where $\gamma = 1.1615 \pm 0.0020$ is the corresponding critical exponent for SAWs in the bulk.[24]

In a (θ, π)-wedge, Cardy[25] has calculated first order ε-expansions for the N-vector model from which one can derive[21]

$$\gamma_1(\theta, \pi) = 1 - \frac{\lambda}{2} - \left(\frac{N+2}{N+8}\right)\left(\frac{\lambda^2 - 12\lambda - 1}{24\lambda}\right)\varepsilon \tag{13}$$

where $\lambda = \pi/\theta$. Our case corresponds to setting $N = 0$ and $\varepsilon = 1$. Guttmann and Torrie[21] have tested Cardy's result by using series analysis techniques on exact enumeration data. They find

$$\gamma_1(\theta, \pi) = \nu(y_0 + y_2 + 2 - d) \tag{14}$$

where, on the basis of RG estimates, the bulk scaling index is

$$y_0 = 2.488 \pm 0.004 \tag{15}$$

and, on the basis of series estimates, they conjecture for the edge scaling index

$$y_2 = \tfrac{1}{2} - (0.847 \pm 0.017)(\pi/\theta) \tag{16}$$

In Table I, our Monte Carlo estimates for γ_1 in (θ, π)-wedges are compared with the direct series estimates of Guttmann and Torrie, values derived from (14)–(16) using $\nu = 0.588 \pm 0.0015$[24] and with values calculated from Cardy's ε-expansion.

Guttmann and Torrie's conjectured form for $\gamma_1(\theta, \pi)$ is linear in λ ($= \pi/\theta$). They conjectured a similar form in two dimensions, a result which conformal invariance arguments subsequently proved to be exact.[15] We find that such a form, with very similar constants, fits our data very well also. Furthermore, we have found that this form can be successfully extended, in a variety of ways, to a corresponding form for $\gamma_1(\theta, \phi)$.

However, in the absence of any firm theoretical basis for the angular dependence of critical exponents (or amplitudes) for either SAWs or stars, we refrain from presenting such fits.

We see that our estimates of $\gamma_1(\theta, \pi)$ in Table I decrease as θ decreases, a result which reflects the increasing confinement that occurs as θ decreases. There is a large difference between γ_1 for the two realizations of an 1/8-space, namely $\gamma_1(\pi/4, \pi) \simeq -0.76$ and $\gamma_1(\pi/2, \pi/2) \simeq -0.283$. Although a SAW in $(\pi/4, \pi)$-wedge has a wide lateral freedom, it is constrained very strongly in its other direction and this presumably accounts for the small value of $\gamma_1(\pi/4, \pi)$. Finally, we note that $\gamma_1(3\pi/2, 3\pi/2) > \gamma_1(3\pi/2, \pi)$, which is consistent with a $(3\pi/2, 3\pi/2)$-wedge corresponding to a 7/8-space and a $(3\pi/2, \pi)$-wedge corresponding to a 3/4-space.

We have also estimated the mean square radius of gyration (about the center of mass) and the mean square end-to-end length of a SAW in a (θ, ϕ)-wedge. We expect that

$$\langle S_N^2(\theta, \phi; 1)\rangle = A(\theta, \phi; 1)\, N^{2\nu}[1 + a(\theta, \phi; 1)\, N^{-\Delta} + \cdots] \tag{17}$$

and

$$\langle R_n^2(\theta, \phi; 1)\rangle = B(\theta, \phi; 1)\, n^{2\nu}[1 + b(\theta, \phi; 1)\, n^{-\Delta} + \cdots] \tag{18}$$

respectively, where in general $N = nf + 1$, ν is about 0.588, and Δ is about 0.47, as for the parent lattice.[24] If we plot $\ln\langle S_N^2(\theta, \phi; 1)\rangle$ against $\ln N$ and $\ln\langle R_n^2(\theta, \phi; 1)\rangle$ against $\ln n$, we obtain a set of parallel straight lines for the various (θ, ϕ)-wedges. This strongly suggests that the exponent ν is independent of (θ, ϕ) and we assume the value $\nu = 0.588$, which is consistent with the slope of the log–log plots.

With this assumption about the value of the exponent ν, we have plotted $\langle S_N^2(\theta, \phi; 1)\rangle/N^{1.176}$ against $N^{-0.47}$ and $\langle R_n^2(\theta, \phi; 1)\rangle/n^{1.176}$ against $n^{-0.47}$, and our estimates of $A(\theta, \phi; 1)$ and $B(\theta, \phi; 1)$ are given in Table II. Sample plots are presented in Fig. 2.

These results for the walk dimensions reflect the effects of the wedge surfaces in exactly the same way as did $\gamma_1(\theta, \phi)$. For example, to within the quoted uncertainties, $A(\theta, \pi; 1)$ and $B(\theta, \pi; 1)$ both increase as θ decreases, since the increasing confinement means that the walk is more extended and hence has a larger radius of gyration. Again the SAW is more extended in the $(\pi/4, \pi)$-wedge than it is in the octant, a conclusion consistent with the relative values of γ_1.

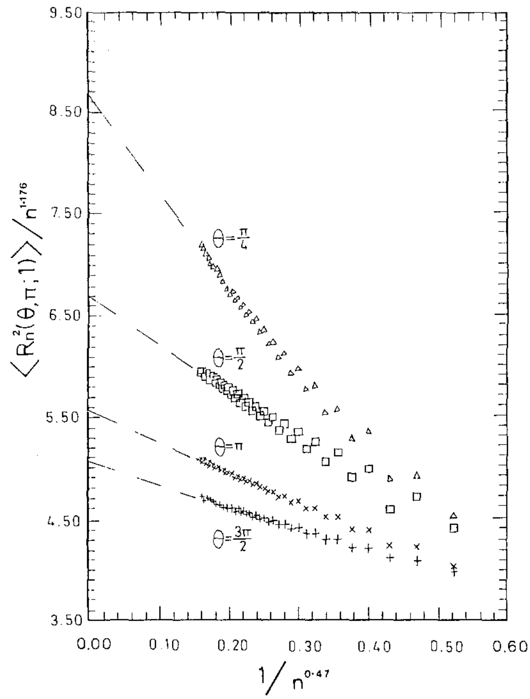

Fig. 2. Monte Carlo estimates of $B(\theta, \pi; 1)$ for the diamond lattice. The largest error bars are about equal to the size of the symbols. To obtain the tabulated estimates, the intercepts should be divided by 3, since we have adopted the usual convention that the norm of a diamond lattice vector is $\sqrt{3}$.

Table I. Values of $\gamma_1(\theta, \phi)$

(θ, ϕ)	Present work	Series estimates[21]	Conjectured[14-16]	ε-Expansion[13]
$(3\pi/2, 3\pi/2)$	0.991 ± 0.011	—	—	—
$(3\pi/2, \pi)$	0.886 ± 0.006	—	0.840 ± 0.015	0.800...
(π, π)	0.703 ± 0.005	0.676 ± 0.009	0.670 ± 0.015	0.625
$(\pi/2, \pi)$	0.220 ± 0.005	0.16 ± 0.03	0.17 ± 0.03	0.109...
$(\pi/4, \pi)$	-0.76 ± 0.06	-0.9 ± 0.2	-0.82 ± 0.05	-0.914...
$(\pi/2, \pi/2)$	-0.283 ± 0.007	—	—	—

4. MONTE CARLO CALCULATIONS FOR STARS

We now turn to consider uniform f-stars ($f > 1$) attached by the vertex of degree f. Data have been obtained for three types of wedge, namely, the (π, π)-wedge ($2 \leqslant f \leqslant 5$), the $(\pi/2, \pi)$-wedge ($f = 2, 3$), and the $(\pi/2, \pi/2)$-wedge ($f = 2$). Estimates of $\gamma_f(\theta, \phi) - 1$ were made by extrapolating plots of $\ln[s_n(\theta, \phi; f)/\mu^{nf}]/\ln nf$ against $1/\ln nf$. Typical plots are shown in Fig. 1 and our best estimates are summarized in Table III. The results for $f = 1$ are reproduced from Table I and are included for comparison.

As expected, for a given wedge, γ_f decreases as f increases, reflecting the increasing interference between the branches. For the (π, π)-wedge, we have enough data to fit the f dependence of $\gamma_f(\pi, \pi)$ by a low-order polynomial. In two-dimensional wedges, the exact result of Duplantier and Saleur[15] is quadratic in f and the first-order ε-expansion[26] is also quadratic,

$$\gamma_f(\pi, \pi) = 1 - \tfrac{1}{2}f - \tfrac{1}{16}f(f-3)\,\varepsilon + O(\varepsilon^2) \tag{19}$$

Assuming a quadratic in f for $\gamma_f(\pi, \pi)$, we find

$$\gamma_f(\pi, \pi) = 1.0513 - 0.27263f - 0.07571f^2 \tag{20}$$

which should be compared with the result obtained by truncating the expansion in (19) and setting $\varepsilon = 1$, namely

$$\gamma_f(\pi, \pi) = 1 - 0.3125f - 0.0625f^2 \tag{21}$$

If we assume a cubic in f, we find that the coefficient of f^3 is of order 10^{-4}. Values of $\gamma_f(\pi, \pi)$ calculated from (20) and (21) are compared in Table III with the corresponding Monte Carlo estimates. Clearly, (20) provides an excellent empirical form for $\gamma_f(\pi, \pi)$.

Table II. Estimates of $A(\theta, \phi; 1)$ and $B(\theta, \phi; 1)$

	$A(\theta, \phi; 1)$		$B(\theta, \phi; 1)$	
(θ, ϕ)	D	SC	D	SC
$(3\pi/2, 3\pi/2)$	0.235 ± 0.002	0.189 ± 0.003	1.597 ± 0.003	1.30 ± 0.02
$(3\pi/2, \pi)$	0.238 ± 0.003	0.205 ± 0.010	1.68 ± 0.01	1.37 ± 0.02
(π, π)	0.241 ± 0.002	0.198 ± 0.004	1.85 ± 0.01	1.49 ± 0.02
$(\pi/2, \pi)$	0.250 ± 0.002	0.208 ± 0.003	2.23 ± 0.02	1.85 ± 0.05
$(\pi/4, \pi)$	0.288 ± 0.002	0.245 ± 0.006	2.88 ± 0.05	2.48 ± 0.08
$(\pi/2, \pi/2)$	0.263 ± 0.002	0.214 ± 0.002	2.63 ± 0.01	1.92 ± 0.06

Table III. Values of $\gamma_f(\theta, \phi)$

	(π, π)			$(\pi/2, \pi)$	$(\pi/2, \pi/2)$
f	Monte Carlo	Quadratic fit[20]	ε-Expansion[21]	Monte Carlo	Monte Carlo
1	0.703 ± 0.005	0.70296	0.625	0.220 ± 0.005	-0.283 ± 0.007
2	0.203 ± 0.007	0.2032	0.125	-0.81 ± 0.03	-1.92 ± 0.03
3	-0.44 ± 0.04	-0.448	-0.5	-2.05 ± 0.05	—
4	-1.26 ± 0.06	-1.251	-1.25	—	—
5	-2.2 ± 0.1[a]	-2.20	-2.125	—	—

[a] Simple cubic lattice only.

We note that the estimates in Table III satisfy the exponent inequalities in (12). As an example, we choose to bound $\gamma_2(\pi, \pi)$ and find

$$2\gamma_1(\pi, \pi) - 1 \geqslant \gamma_2(\pi, \pi) \geqslant 2\gamma_1(\pi/2, \pi) - 1$$

or, substituting numerical values, $0.406 \geqslant \gamma_2(\pi, \pi) \geqslant -0.56$, which is consistent with $\gamma_2(\pi, \pi) = 0.203 \pm 0.007$. Unfortunately, none of the bounds so derived are very useful, since they are all rather weak.

Finally, we have estimated the mean square radius of gyration $\langle S_N^2(\theta, \phi; f) \rangle$ and the mean square end-to-end length $\langle R_n^2(\theta, \phi; f) \rangle$ of a branch of an f-star in a (θ, ϕ)-wedge. We have followed the method described earlier for SAWs ($f = 1$), and have assumed, once again, a value of $\nu = 0.588$ and an asymptotic behavior analogous to that given in (17) and (18). These assumptions are supported by the data. Our estimates of the amplitudes $A(\theta, \phi; f)$ and $B(\theta, \phi; f)$ are given in Tables IV and V, respectively.

As expected (see Table V), increasing f extends the star, as does decreasing the size of the wedge. On the other hand, the mean square

Table IV. Estimates of $A(\theta, \phi; f)$

	(π, π)		$(\pi/2, \pi)$		$(\pi/2, \pi/2)$	
f	D	SC	D	SC	D	SC
1	0.241 ± 0.002	0.198 ± 0.004	0.250 ± 0.002	0.208 ± 0.003	0.263 ± 0.002	0.214 ± 0.002
2	0.204 ± 0.001	0.173 ± 0.004	0.183 ± 0.003	0.157 ± 0.002	0.159 ± 0.002	0.133 ± 0.004
3	0.151 ± 0.001	0.128 ± 0.002	0.140 ± 0.003	0.118 ± 0.003	—	—
4	0.123 ± 0.001	0.101 ± 0.001	—	—	—	—
5	—	0.082 ± 0.001	—	—	—	—

Table V. Estimates of $B(\theta, \phi; f)$

	(π, π)		$(\pi/2, \pi)$		$(\pi/2, \pi/2)$	
f	D	SC	D	SC	D	SC
1	1.85 ± 0.01	1.49 ± 0.02	2.23 ± 0.02	1.85 ± 0.05	2.63 ± 0.01	1.92 ± 0.06
2	1.95 ± 0.01	1.575 ± 0.005	2.30 ± 0.02	1.93 ± 0.03	2.80 ± 0.03	2.36 ± 0.04
3	2.00 ± 0.01	1.66 ± 0.01	2.45 ± 0.02	1.98 ± 0.02	—	—
4	2.07 ± 0.01	1.693 ± 0.003	—	—	—	—
5	—	1.775 ± 0.005	—	—	—	—

radius of gyration (see Table IV) decreases as f increases and, for $f > 1$, decreases as the wedge size decreases. The first part of this observation can be understood by remembering that the radii of gyration of different f-stars are compared at fixed N, whereas comparison of their end-to-end lengths is done at fixed n. To understand the second part of the observation, we must remember that the radius of gyration is calculated about the center of mass of the f-star. Hence, as the wedge size decreases, the branches are forced closer to their center of mass even though their lengths increase, and the result of this competition is a decrease in the radius of gyration. We note (see Tables II and V) that for a given wedge the branches of an f-star ($f \geqslant 1$) are longer on the diamond lattice than on the simple cubic lattice. This reflects the more open structure and smaller coordination number of the diamond lattice.

In Tables VI and VII, we give the amplitude ratios $A(\theta, \phi; f)/A(\theta, \phi; 1)$ and $B(\theta, \phi; f)/B(\theta, \phi; 1)$, respectively, which are expected to be lattice-independent quantities[9,27,28] and could in principle be compared with experimental values. In related work,[9] we found that the estimated values of amplitude ratios were essentially independent of the assumed

Table VI. Estimates of Amplitude Ratios $A(\theta, \phi; f)/A(\theta, \phi; 1)$

	(π, π)		$(\pi/2, \pi)$		$(\pi/2, \pi/2)$	
f	D	SC	D	SC	D	SC
1	1	1	1	1	1	1
2	0.85 ± 0.02	0.87 ± 0.04	0.73 ± 0.02	0.76 ± 0.03	0.61 ± 0.02	0.62 ± 0.03
3	0.63 ± 0.01	0.65 ± 0.03	0.56 ± 0.02	0.57 ± 0.03	—	—
4	0.51 ± 0.01	0.51 ± 0.02	—	—	—	—
5	—	0.41 ± 0.02	—	—	—	—

Table VII. Estimates of Amplitude Ratios $B(\theta, \phi; f)/B(\theta, \phi; 1)$

	(π, π)		$(\pi/2, \pi)$		$(\pi/2, \pi/2)$	
f	D	SC	D	SC	D	SC
1	1	1	1	1	1	1
2	1.05 ± 0.02	1.06 ± 0.02	1.03 ± 0.02	1.04 ± 0.05	1.07 ± 0.02	1.23 ± 0.06
3	1.08 ± 0.01	1.11 ± 0.03	1.10 ± 0.02	1.07 ± 0.04	—	—
4	1.12 ± 0.01	1.14 ± 0.02	—	—	—	—
5	—	1.19 ± 0.02	—	—	—	—

value of ν in the range 0.58–0.6. It appears that, to within the numerical uncertainties, our results are—with one exception—consistent with lattice independence. The exception probably results from us underestimating the uncertainties.

ACKNOWLEDGMENTS

D.S.G. wishes to dedicate this article to Prof. C. Domb on the occasion of his retirement and to take this opportunity to thank him, not only for introducing him to statistical physics, but also for numerous helpful discussions and kindnesses over many years.

We have benefited from discussions and correspondence with S.G. Whittington and B. Duplantier. D.S.G. is grateful to NATO (grant RG85/0067) for partial financial support and S.A.C. thanks the SERC for the award of a research studentship.

REFERENCES

1. M. Daoud and J. P. Cotton, *J. Phys.* (Paris) **43**:531 (1982).
2. T. M. Birshtein and E. B. Zhulina, *Polymer* **25**:1453 (1984).
3. A. Miyake and K. F. Freed, *Macromolecules* **16**:1228 (1983).
4. A. Miyake and K. F. Freed, *Macromolecules* **17**:678 (1984).
5. C. H. Vlahos and M. K. Kosmas, *Polymer* **25**:1607 (1984).
6. B. Duplantier, *Phys. Rev. Lett.* **57**:941 (1986).
7. J. E. G. Lipson, S. G. Whittington, M. K. Wilkinson, J. L. Martin, and D. S. Gaunt, *J. Phys. A: Math. Gen.* **18**:L469 (1985).
8. M. K. Wilkinson, D. S. Gaunt, J. E. G. Lipson, and S. G. Whittington, *J. Phys. A: Math. Gen.* **19**:789 (1986).
9. S. G. Whittington, J. E. G. Lipson, M. K. Wilkinson, and D. S. Gaunt, *Macromolecules* **19**:1241 (1986).
10. A. J. Barrett and D. L. Tremain, *Macromolecules* **20**:1687 (1987).

11. M. N. Chee and S. G. Whittington, *J. Phys. A: Math. Gen.* **20**:4915 (1987).
12. C. E. Soteros and S. G. Whittington, *J. Phys. A: Math. Gen.* **21**:L857 (1988).
13. S. A. Colby, D. S. Gaunt, G. M. Torrie, and S. G. Whittington, *J. Phys. A: Math. Gen.* **20**:L515 (1987).
14. J. M. Hammersley and S. G. Whittington, *J. Phys. A: Math. Gen.* **18**:101 (1985).
15. B. Duplantier and H. Saleur, *Phys. Rev. Lett.* **57**:3179 (1986).
16. M. N. Rosenbluth and A. W. Rosenbluth, *J. Chem. Phys.* **23**:356 (1955).
17. K. M. Middlemiss and S. G. Whittington, *J. Chem. Phys.* **64**:4684 (1976).
18. L. Ma, K. M. Middlemiss, and S. G. Whittington, *Macromolecules* **10**:1415 (1977).
19. M. N. Barber, A. J. Guttmann, K. M. Middlemiss, G. M. Torrie, and S. G. Whittington, *J. Phys. A: Math. Gen.* **11**:1833 (1978).
20. K. Ohno, Y. Okabe, and A. Morita, *Prog. Theor. Phys.* **71**:714 (1984).
21. A. J. Guttmann and G. M. Torrie, *J. Phys. A: Math. Gen.* **17**:3539 (1984).
22. E. Eisenriegler, K. Kremer, and K. Binder, *J. Chem. Phys.* **77**:6296 (1982).
23. H. W. Diehl and S. Dietrich, *Phys. Lett.* **80A**:408 (1980).
24. J. C. Le Guillou and J. Zinn-Justin, *Phys. Rev. B* **21**:3976 (1980).
25. J. L. Cardy, *J. Phys. A: Math. Gen.* **16**:3617 (1983).
26. K. Ohno and K. Binder, *J. Phys.* (Paris) **49**:1329 (1988).
27. J. E. G. Lipson, D. S. Gaunt, M. K. Wilkinson, and S. G. Whittington, *Macromolecules* **20**:186 (1987).
28. S. G. Whittington, M. K. Kosmas, and D. S. Gaunt, *J. Phys. A: Math. Gen.* **21**:4211 (1988).